Scaling laws for spatiotemporal synchronization and array enhanced stochastic resonance

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Recently, the synchronization and signal processing ability of a locally and linearly coupled array of bistable elements was enhanced by the addition of uncorrelated noise [J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. Lett. **75**, 3 (1995)]. Here, we detail the performance of such an array as a function of both coupling and noise. Simple theoretical arguments, grounded in extensive numerical studies, suggest how to "tune" the array for best synchronization and signal-to-noise ratio. Specifically, we propose that, for large array size N, the optimal coupling scales like N^2 and the optimal noise variance scales like N. This scaling matches the coupling-induced correlation length to the array length and the noise-generated mean hopping time to the modulation period, thereby creating a stochastic resonance in space and time.

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A noisy, periodically modulated, nonlinear system undergoes *stochastic resonance* (SR) when the characteristic time scales of the stochastic forcing and the deterministic forcing match [1]. Recent research has demonstrated that SR can be enhanced by coupling a single stochastic resonator into an array of identical resonators [2–5]. Spatiotemporal synchronization improves the response of an individual resonator to the modulation, as measured by an enhanced signal-to-noise ratio (SNR). The resulting *array enhanced stochastic resonance* (AESR) is characterized by a significantly increased output SNR, compared to a single uncoupled resonator. In this paper, we understand AESR and its attendant spatiotemporal synchronization as a matching of time *and* space scales through the tuning of noise *and* coupling. We document the necessary tuning by extensive numerical simulations and, based on these data, provide theoretical arguments to quantify the scaling of optimal noise and optimal coupling with array size. We propose that, for large array size N, the optimal noise scales as N and the optimal coupling scales as N^2.

We study the same locally and linearly coupled array of periodically forced, bistable dynamic elements (overdamped oscillators, stochastic resonators) as in Ref. [2]. The equations of motion are

$$m\ddot{x}_n+\gamma\dot{x}_n=kx_n-k'x_n^3+\varepsilon(x_{n-1}-x_n)+\varepsilon(x_{n+1}-x_n)+A\sin(2\pi t/T)+\sigma G_n(t)\ , \quad (1)$$

where $n=1,2,\dots,N$, and we impose free boundary conditions. To reduce the dimension of the parameter space, we study the overdamped limit $\gamma\dot{x}\gg m\ddot{x}$. We take $G_n(t)$ to be Gaussian white noise with zero mean and unit power spectral density (PSD). However, in practice, $\sigma G_n(t)$ is band limited with a PSD of height D for frequencies $|f|\le f_N$ and zero beyond. We characterize the noise either by its PSD height or by its mean squared amplitude or *variance* $\sigma^2=2Df_N$. Note that the noise is "local" (uncorrelated from site to site) rather than "global" (correlated from site to site) [3,4].

We numerically integrate the stochastic differential equation (1) using the Euler-Maruyama scheme [6] with a time step $dt=1/(2f_N)=T/8192$ that is half the time step used in Ref. [2]. We Fourier analyze the discretely sampled time series of the middle oscillator in the array to compute a PSD histogram. From this, we compute an SNR, defined here as the ratio of the signal power divided by the noise power in the signal frequency bin, expressed in decibels (dB). Note that the state point of each oscillator moves in a double-welled potential. Unlike Ref. [2], we do not first filter the time series to remove intrawell motion. However, as in [2], we choose our operating regime just below the deterministic switching threshold so that in the absence of noise the oscillator is confined to a single well of the bistable potential, but small noise can induce significant hopping between wells. We obtain similar results for smaller driving amplitudes.

Figure 1 illustrates the spatiotemporal evolution of an array of $N=512$ oscillators, at moderate values of coupling and noise. Time t increases upward; oscillator index n increases from left to right. An oscillator is colored blue if it is in the left well and red if it is in the right well. The saturation of the colors decreases to zero (white) at the center, reflecting the ambiguity of assigning a well to an oscillator on the barrier between the wells. All the os-

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cillators were started in the blue well but with a random spread in their initial positions. A short transient was omitted. The equal amounts of red and blue in the figure reflect the symmetry of the bistable potential.

Figure 2 illustrates the effect of varying the coupling and the noise, this time on the spatiotemporal evolution of a chain of 65 oscillators. In the top panel the periodic forcing is turned off, and in the bottom panel the forcing is turned on. Consider first the case of no forcing. Let λ be the spatial scale or correlation length, and let τ be the

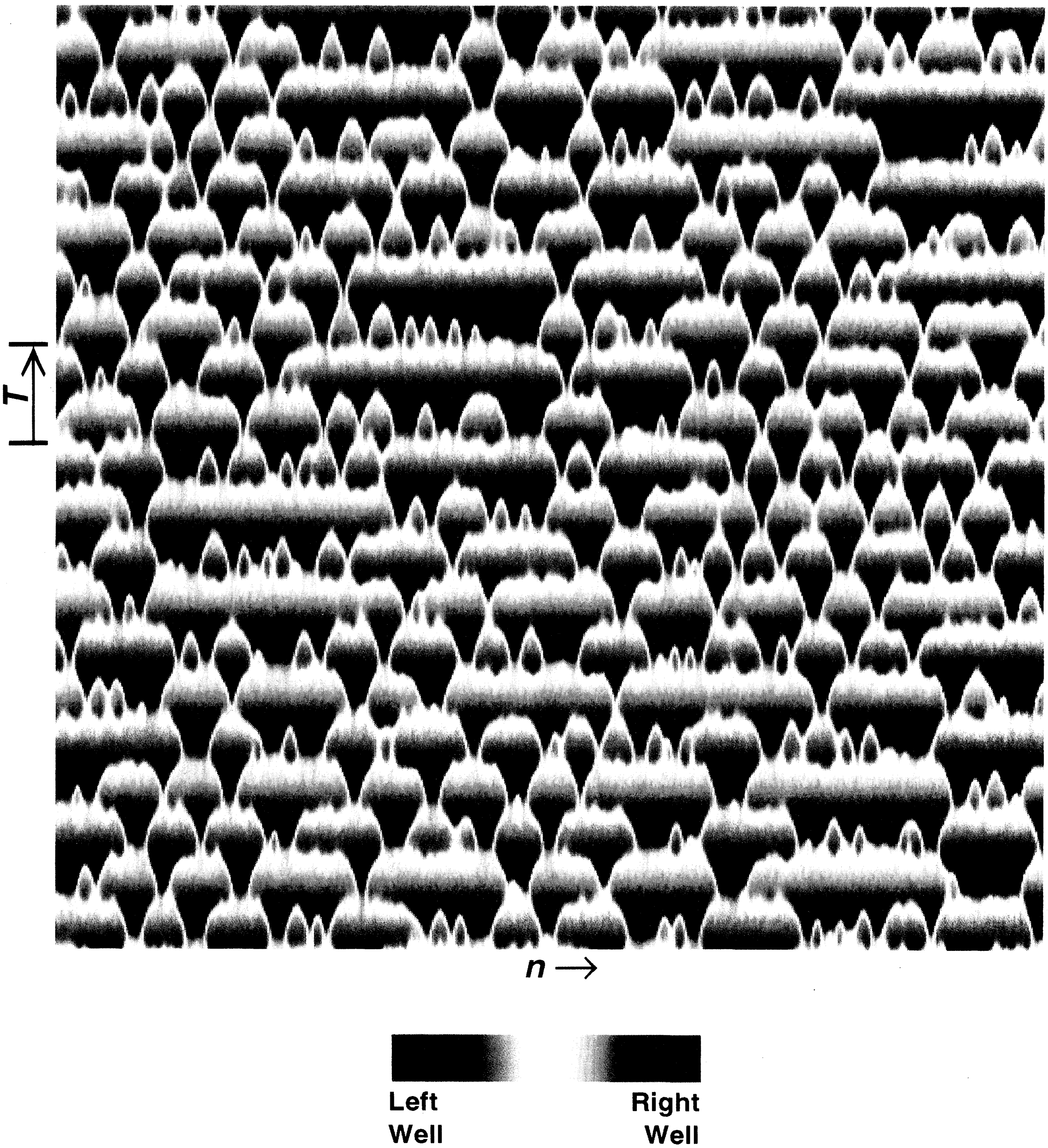

FIG. 1. Time evolution (up) of an array (across) of 512 locally and linearly coupled, bistable nonlinear oscillators. Oscillators in the left well are colored blue while oscillators in the right well are colored red. A single forcing period is marked on the vertical scale. Coupling is $\varepsilon=1$, and noise variance is $10\log_{10}\sigma^2=15$ dB. Other parameters: $k=2.1078$, $k'=1.4706$, $A=1.3039$, $f=1/T=0.1162$.

time scale for hopping. The "area" of a typical red (or blue) feature is $\lambda\tau$. Clearly, λ must increase from 1 to N as coupling ε increases from 0 to infinity, but τ must *decrease* from infinity to dt as noise σ^2 increases from 0 to infinity. Thus, λ and τ are both large when the coupling is large and the noise is small; conversely, λ and τ are both small when the coupling is small and the noise is large. Consequently, the bottom right of noise versus

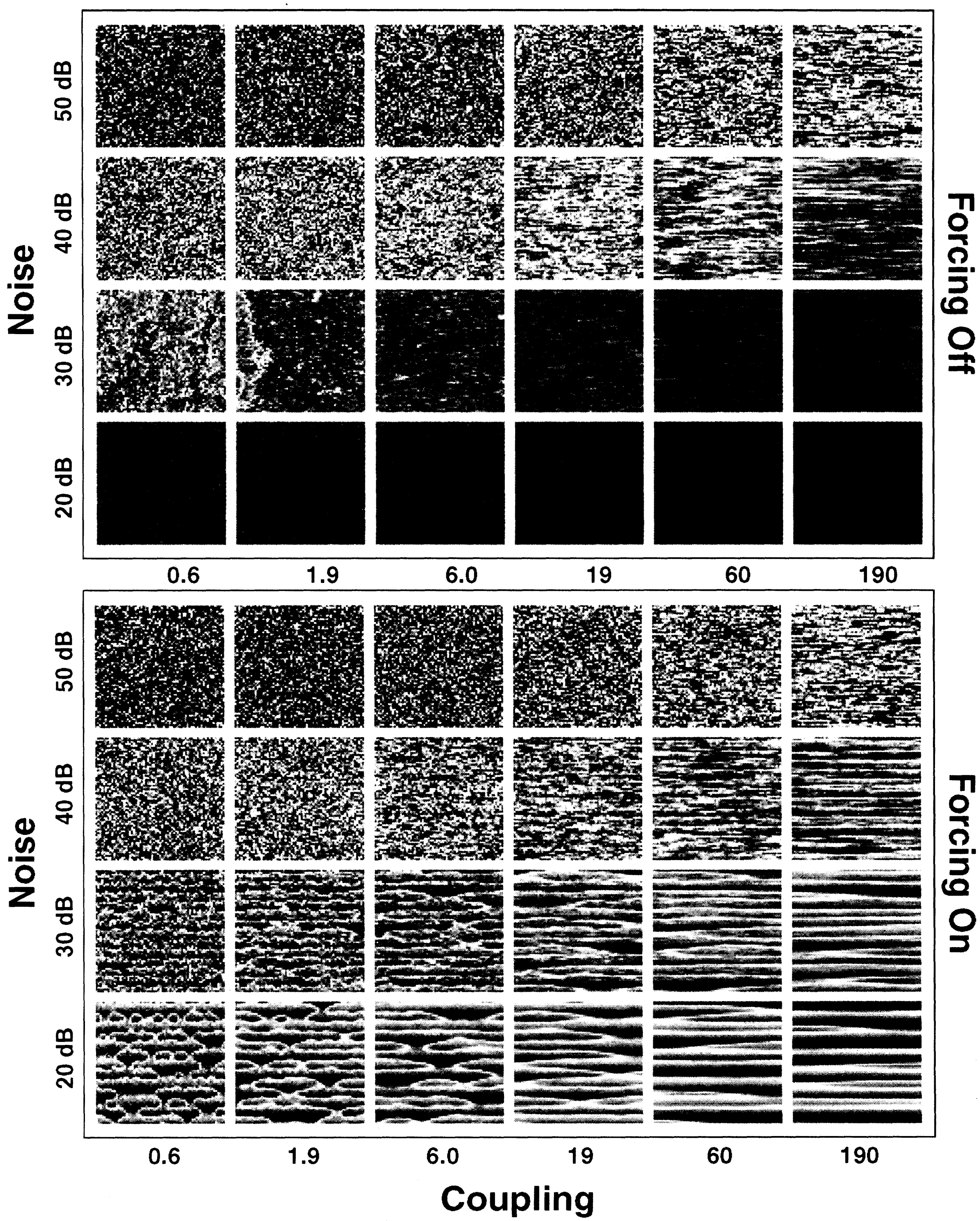

FIG. 2. Effect of varying coupling and noise on the spatiotemporal evolution of an array of 65 oscillators. In the top panel the periodic forcing is turned off; in bottom panel the forcing is turned on. Coupling and noise control the space and times scales: small spatiotemporal features at top left and large spatiotemporal features at bottom right. All oscillators were started in random positions in the blue well. Within each colored square, time increases upward and position along the array varies horizontally. Parameters are the same as in Fig. 1.

coupling space is dominated by large spatiotemporal features while the top left is dominated by small spatiotemporal features.

Now turn on the forcing. The interplay of noise and coupling is captured in the following symmetric pair of observations. For a given noise, if the coupling is suboptimal (in terms of maximizing output SNR), adjacent oscillators are likely to be forced in opposite wells; if the coupling is superoptimal, the entire chain is likely to spend consecutive forcing periods trapped in the same

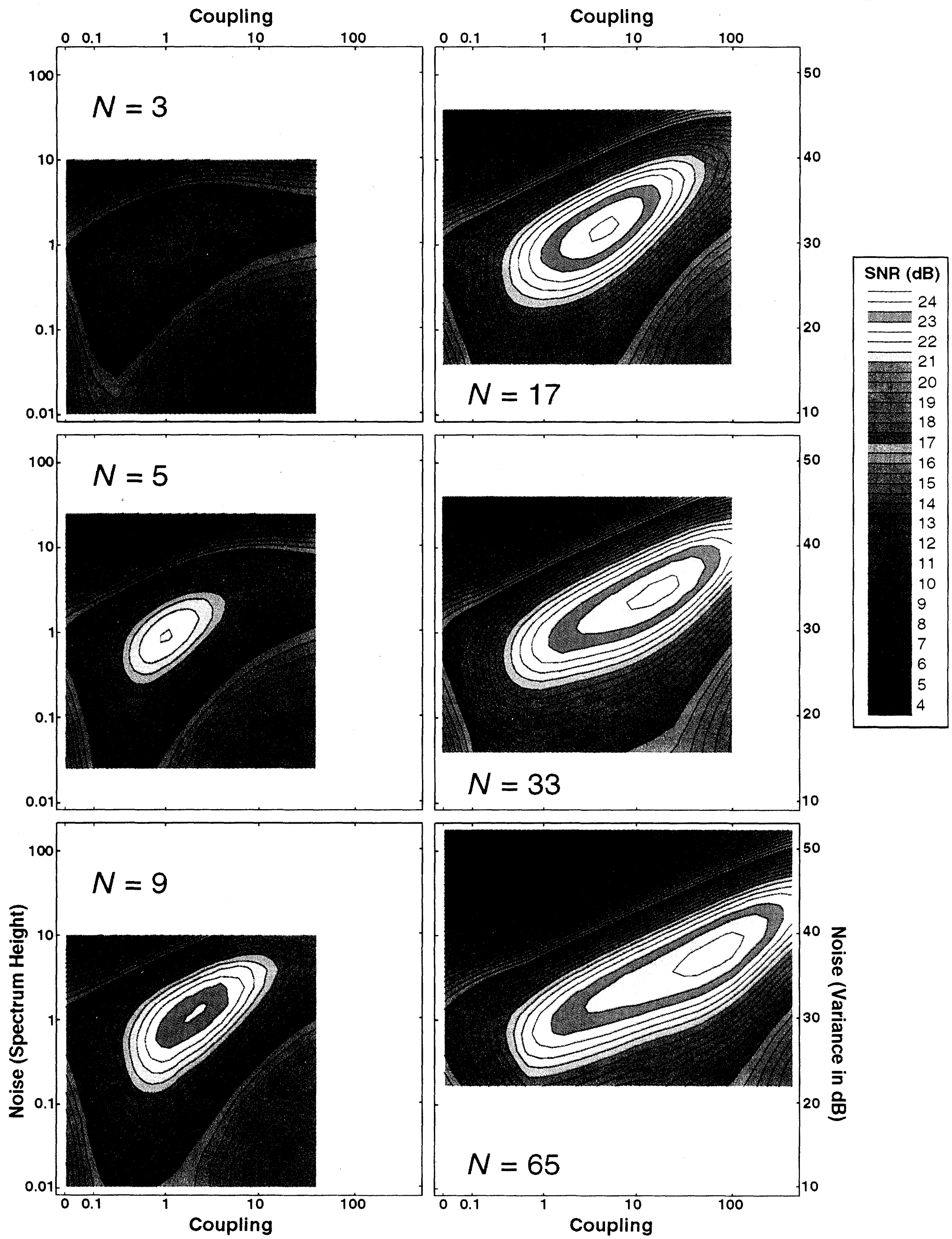

FIG. 3. Contours of SNR plotted against a "tuning space" of coupling vs noise. Single uncoupled oscillator (left edge) outperforms the "water"-colored contours while the "island"-colored contours outperform the uncoupled oscillator. Parameters are the same as in Fig. 1.

well. (Superoptimal coupling makes the chain move more like a single element, which underperforms an array.) For a given coupling, if the noise is suboptimal, the entire chain is likely to spend consecutive forcing periods trapped in the same well; if the noise is superoptimal, adjacent oscillators are likely to be forced in opposite wells. Spatiotemporal synchronization (best SNR, two hops per forcing period) involves balancing these two factors so that $\lambda \sim N$ and $\tau \sim T$. This balance exists near the upper right of the top panel of Fig. 2, where there are large patches of blue and red. When the periodic forcing is turned on in the bottom panel, these patches are replaced by horizontal blue and red bands signifying synchronous hopping.

Figure 3 illustrates the SNR of the middle oscillator as contours plotted against noise and coupling, for six different arrays sizes: 3, 5, 9, 17, 33, and 65. (These sizes were chosen to be of the form 2^n+1 so as to provide arrays with middle elements, and data evenly spaced on a logarithmic scale.) The noise scales are logarithmic while the couplings scales are *almost* logarithmic: in place of a $\ln(\varepsilon)$ scale, we use $\ln(0.1+\varepsilon)$, allowing us to include, for comparison, the uncoupled $\varepsilon=0$ case at the left edge of the plots. A single uncoupled oscillator outperforms the region of "tuning space" indicated by blue "water" colors. Conversely, the region indicated by "island" colors outperforms a single uncoupled oscillator. Note how the best SNR shifts to higher coupling and higher noise as it saturates at about 25 dB. The best a single uncoupled oscillator can do is about 17 dB.

Because of the computationally intensive PSD averaging needed to form smooth contours, Fig. 3 represents thousands of node hours of parallel supercomputer time (two Intel PARAGON parallel supercomputers were used). A more economical way of locating the best SNR would be to compute instead some measure of synchronization, such as the occupancy function introduced in Ref. [2].

Figure 4 illustrates the scaling of the optimal noise and optimal coupling with array size. Here, we plot the normalized coupling ε/N^2 and normalized noise D/N. For large N, these curves appear to asymptote to constant values, as they should if the proposed scaling is correct. Furthermore, they mirror the behavior of the best SNR curve as it saturates at about 25 dB. The insets illustrate $\sqrt{\varepsilon}$ versus N and D versus N. The insets illustrate $\sqrt{\varepsilon}$ versus N and D vs N. They suggest that the scaling is approximately true for all N. However, a log-log plot reveals significant nonlinearity for small N.

The optimal coupling scales as N^2 for the following reason. Best SNR is characterized by synchronous hopping of the chain [3,4]. The optimal coupling facilitates

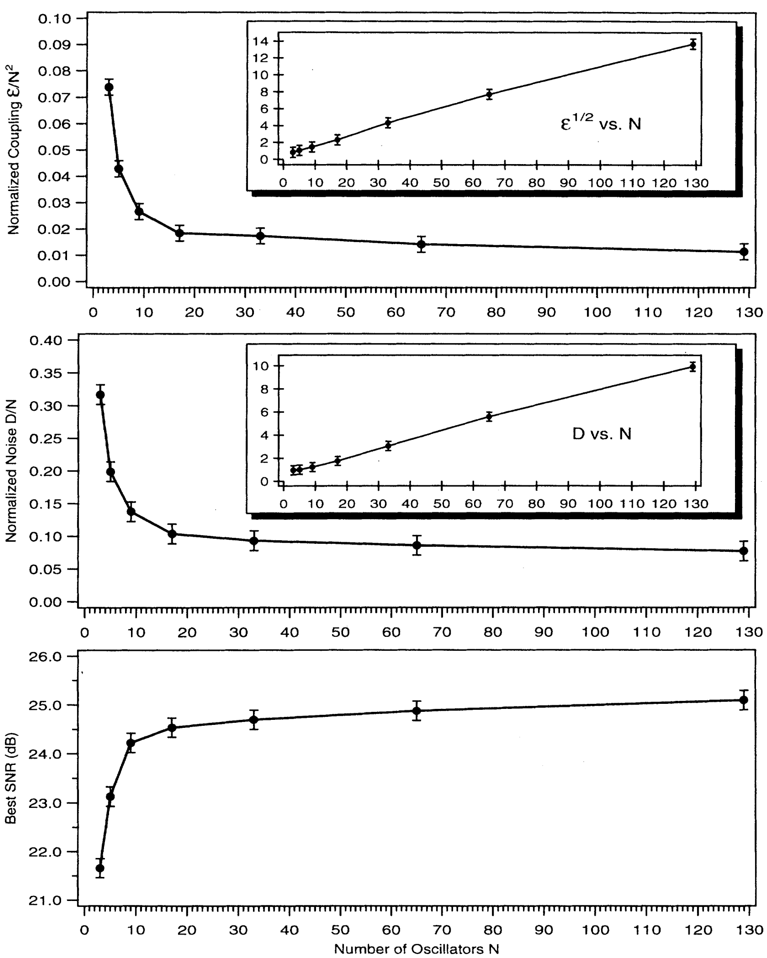

FIG. 4. Scaling of optimal coupling, optimal noise, and best SNR. The normalized coupling and noise curves appear to asymptote to constant values, as does the best SNR curve, thereby supporting the scaling hypotheses. The insets illustrate $\sqrt{\varepsilon}$ vs N and D vs N. Parameters are the same as in Fig. 1.

this hopping by binding the oscillators together tightly enough so that the average size of spatial features in the array, the correlation length, is on the order of N, the size of the array. How does correlation length scale with the coupling? If ε is sufficiently large, we can neglect the potential term in Eq. (1), and if we turn off the noise, the equations of motion become

$$m\ddot{x}_n = \varepsilon\nabla^2 x_n + \gamma\dot{x}_n + A\sin(2\pi t/T) , \tag{2}$$

where ∇^2 is the discrete Laplacian. We recognize this as a forced, damped wave equation with a wave velocity proportional to $\sqrt{\varepsilon}$. Since the correlation length is the length of the array which self-communicates, and since information travels at the wave velocity, the correlation length must be proportional to $\sqrt{\varepsilon}$ [7]. The *optimal* coupling ε_0 is determined by $\sqrt{\varepsilon_0} \sim N$ or $\varepsilon_0 \sim N^2$.

The optimal noise variance scales as N for the following reason. The optimal noise facilitates synchronized motion by causing hopping of the entire array, bound by the optimal coupling to a correlation length of N, twice every forcing period. The noise has zero mean; it is fluctuations from the mean that cause hopping. The larger the array, the larger the needed fluctuation. In order to expect such a fluctuation twice per forcing period, the variance of the noise must increase as N increases. How should the variance scale with array size? Let $\wp(f \geq f_{\min}|\sigma^2)$ be the probability per unit time that a single uncoupled oscillator receives the minimum impulse $f_{\min}$ needed to hop over the barrier, given that the noise distribution has variance σ^2. A coupled and coherent array of size N, responding as a unit, will require an impulse N times as great. However, for uncorrelated Gaussian noise,

$$\wp\left[\sum_{n=1}^{N} f_n \geq N f_{\min}|N\sigma^2\right] = \wp(f \geq \sqrt{N} f_{\min}|N\sigma^2) = \wp(f \geq f_{\min}|\sigma^2) . \tag{3}$$

The first equality in Eq. (3) follows from the fact that the standard deviation of the sum of N independent identically distributed Gaussian random variables is $\sqrt{N}$ times as large as their individual standard deviations. The second equality follows because variance is the square root of the standard deviation. Thus, in order to expect the necessary fluctuation twice per forcing period, the *optimal* noise variance σ_0^2 and PSD height D_0 must scale like $D_0 \propto \sigma_0^2 \sim N$. A simple ϕ^4 theory of kink-antikink nucleation in a lattice yields the identical scaling behavior [8].

In summary, we have argued that to tune ever larger arrays of locally and linearly coupled bistable oscillators to peak performance, as measured by signal processing and synchronization, the optimal coupling must scale like N^2 and the optimal noise variance must scale like N, for large N. If this scaling is maintained, the coupling-induced correlation length will match the length of the array and the noise-generated mean hopping time will match (half of) the forcing period, and a spatiotemporal or *space-time stochastic resonance* will result.

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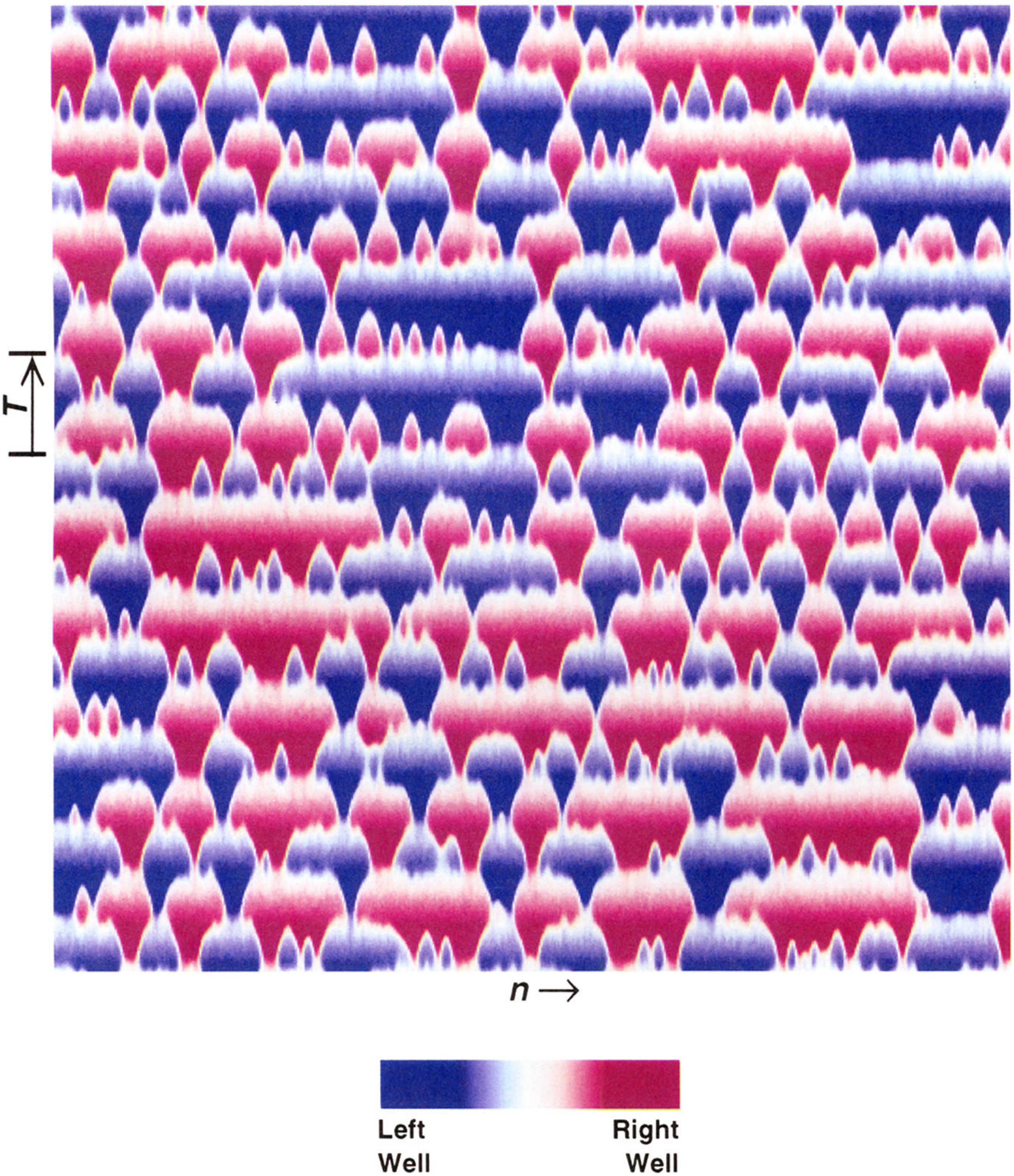

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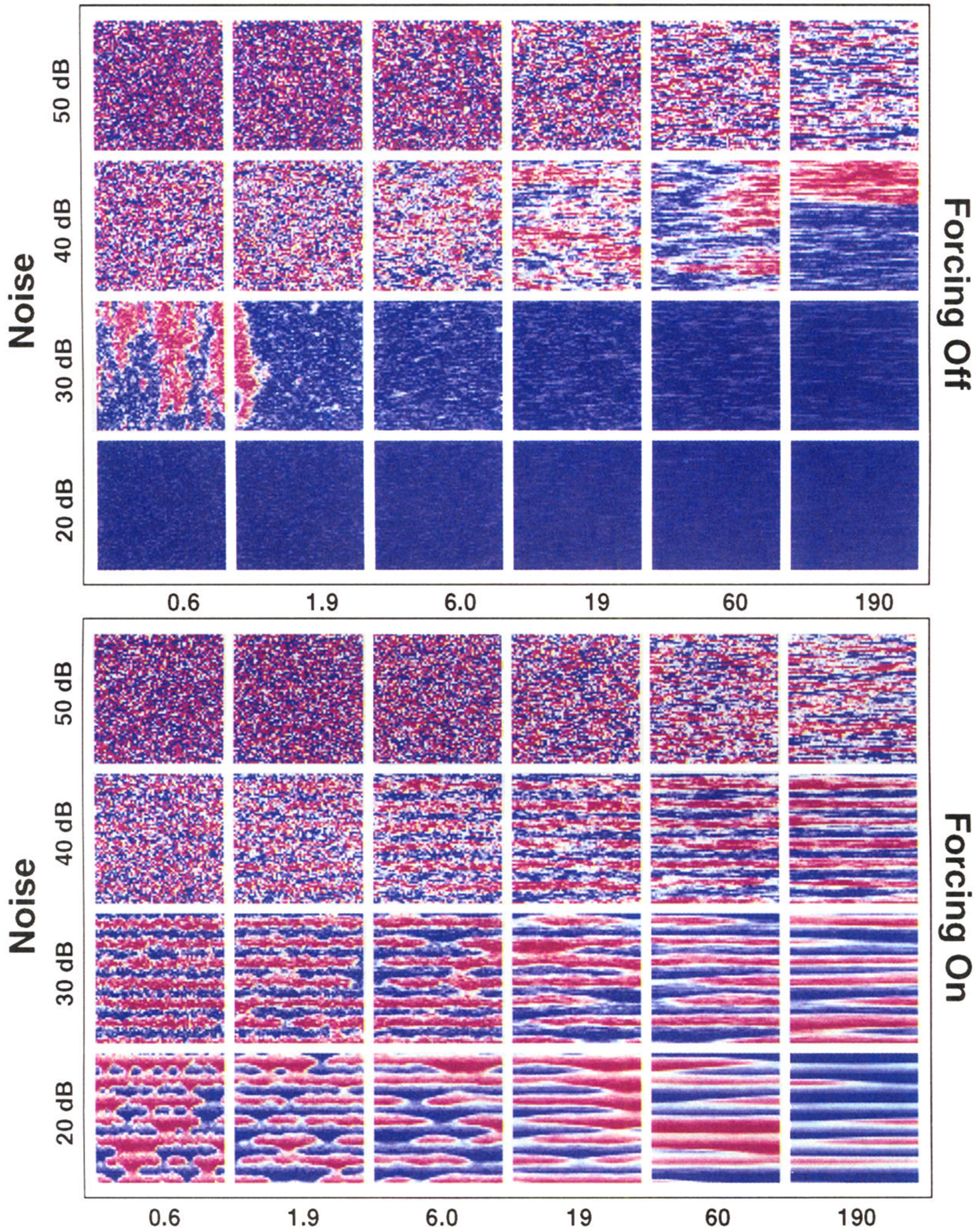

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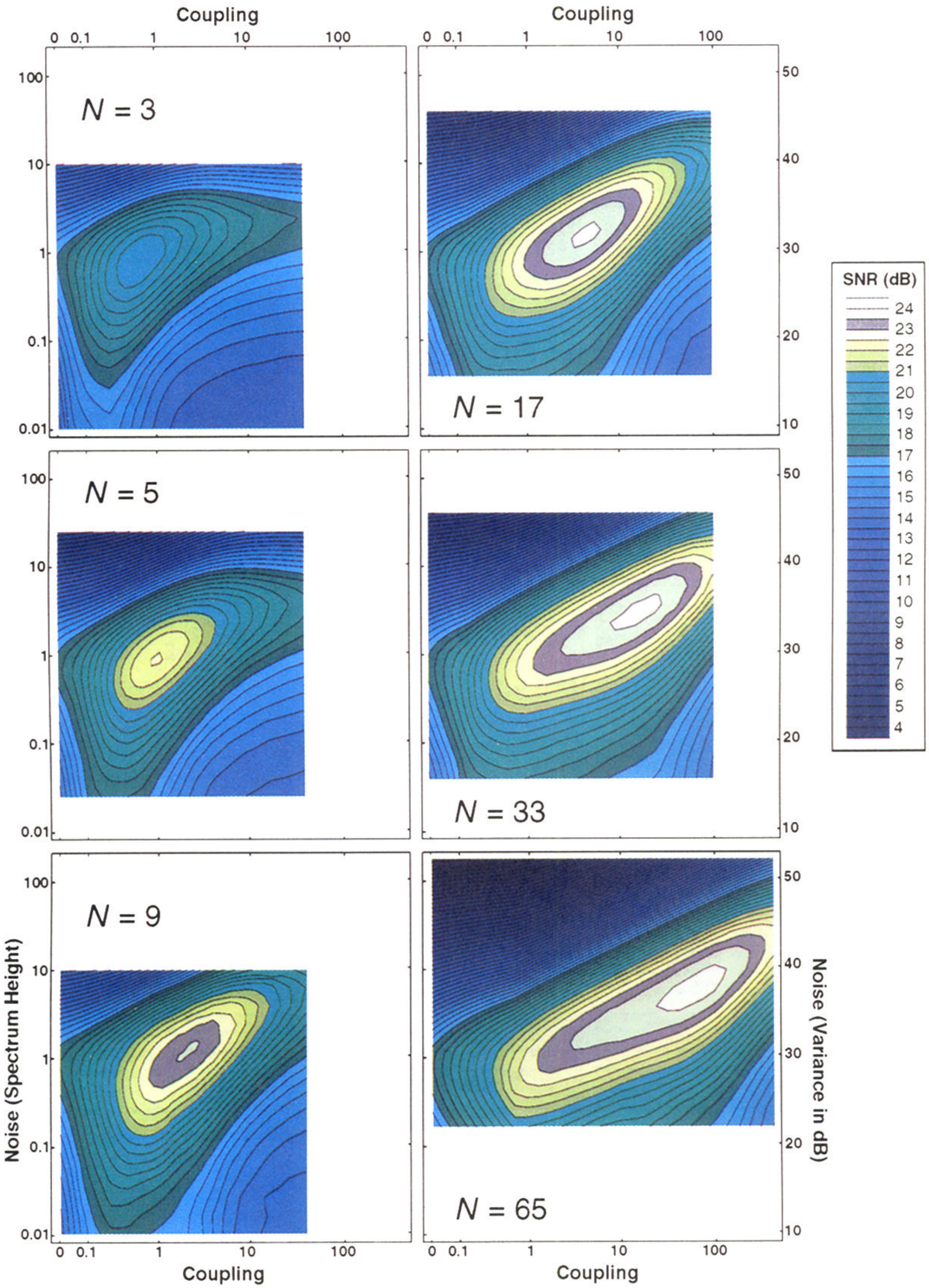

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